

STUDY ON DIOPHANTINE EQUATION AND ITS REAL-LIFE APPLICATIONS

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ABSTRACT

Polynomial equations with integer solutions, or diophantine equations, are a set of equations with wide applications in many different fields. This essay investigates the practical uses of Diophantine equations in areas like number theory, network traffic flow, corporate investment, and chemical equation balance. It also looks at the historical evolution of these equations (see [4-5]). We show how useful Diophantine equations are for solving real-world issues requiring integer solutions by examining both linear and non-linear versions of the equation. The importance of these equations for both theoretical mathematics and practical applications is demonstrated by highlighting well-known outcomes such Pythagorean triples and Fermat's Last Theorem. In order to address difficult problems in real-world circumstances and simplify them, the study intends to emphasize the significance of Diophantine equations.

Introduction

An indeterminate polynomial equation with just integer variables is called a Diophantine equation. Diophantine problems entail finding integers that operate appropriately for all equations and have fewer equations with unknown variables. They bear the name of Diophantine, an Alexandrian mathematician from the Hellenistic era (250AD). Diophantine Analysis is the mathematical study of Diophantine problems. One accomplishment of the twentieth century was the development of generic theories of Diophantine equations. Diophantine equations can have an unlimited number of solutions, a finite number of solutions, or none at all. A polynomial equation with an integral coefficient that can be solved in whole numbers is called the Diophantine equation (see [1-3,6,7]). It started because additional equations with one or more unknowns that could only have integral solutions needed to be looked into.

Attributes of Diophantine Equation

Stripe 1.

When $x = \frac{b}{a}$ must be integral, the Diophantine Equation is commonly expressed as a Linear Diophantine Equation with one variable, $ax = b$.

To illustrate, if $\frac{3}{5}$ th of a number is 4 more than $\frac{1}{3}$ rd of the number then what is the integral?

$$\frac{3x}{5} - \frac{x}{3} = 4 \Rightarrow 30 \quad (\text{integral})$$

Stripe 2.

A Diophantine Equation in the form $ax + by = c$ with two unknowns, x and y. If the preceding equation is satisfied by $x = x_0, y = y_0$, then we write the solution as follows, (x_0, y_0) with the requirement that the solutions be limited to integers.

To illustrate, $3x + y = 10$ is a linear Diophantine equation with $x = 3, y = 1$ as its solution. Since, $(3 \times 3) + (1 \times 1) = 10$.

A common condition for the linear Diophantine equations solvability is provided by the following theorem.

Theorem 1: In integers, the equation $ax + by = c$ can be solved iff $d \mid c$, where $d = \gcd(a, b)$.

Proof:

Given that $(a, b) = d$ and the assumption that $d \mid c$, there exist integers x_1 and y_1 such that $ax_1 + by_1 = d$.

Multiplying c/d on both side

$$a\left(\frac{c}{d}\right)x_1 + b\left(\frac{c}{d}\right)y_1 = c$$

$$\Rightarrow x = \left(\frac{c}{d}\right)x_1$$

$$y = \left(\frac{c}{d}\right)y_1$$

In contrast, assume that the answer to the above equation is (x_1, y_1) . Then, $ax_1 + by_1 = c$. Now d splits a and $b \Rightarrow d \mid c$.

Note: Diophantine equations can be enlarged in both linear and nonlinear ways for finite and infinitely many variables.

Real-life Application of Diophantine Equation

Problem 1 on Age: One year ago, Sonam finished her first year of nursery school, making that $1/8^{\text{th}}$ and $1/9^{\text{th}}$ of her life at that time. She has been in primary school for 4 years. What age is she now?

$$\frac{P-1}{8} + \frac{P+1}{9} = 4 \Rightarrow P = 17 \quad (\text{Present age})$$

Problem 2: Laxmi lived through her childhood for $1/8^{\text{th}}$ years, then experienced growth after $1/14^{\text{th}}$ more years, marriage after $1/9^{\text{th}}$ more years, and the birth of her daughter after 7 more years. The mother passed away 6 years after her daughter, and the daughter survived to become half the age of her mother.

$$\frac{a}{8} + \frac{a}{14} + \frac{a}{9} + 7 + \frac{a}{2} + 6 = a \Rightarrow 68 (\text{her age})$$

Note: The system of linear equations is identical to the system that is created when numerous simultaneous linear Diophantine equations are presented in multiple variables.

Problem on business: Ashok divided his money between 0.05 Producer A and 0.1 Producer B. In the first year, he received 5000 rupees in interest income. His income in the second year rises by rupee 3000 if he increases his investment by 0.05 in Producer B and by 0.5 in Producer A. Track down his original investments.

Solution:

Considering he has invested Rs x and Rs y in producers A and B, respectively, then

In first year,

$$0.05x + 0.1y = 5000$$

In second year,

$$0.5(0.05x) + 0.05(0.1y) = 8000$$

$$\Rightarrow 0.025x + 0.005y = 8000$$

as $AX = B$, we can solve these equations.

$$\begin{bmatrix} 0.05 & 0.1 \\ 0.025 & 0.005 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5000 \\ 8000 \end{bmatrix}$$

Since, $|A| = -0.00225 \neq 0$, so A is invertible, Then

$$x = A^{-1}B$$

$$= \frac{1}{-0.00225} \begin{bmatrix} 0.005 & -0.1 \\ -0.025 & 0.05 \end{bmatrix} \begin{bmatrix} 5000 \\ 8000 \end{bmatrix}$$

$$= \frac{1}{-0.00225} \begin{bmatrix} -775 \\ 275 \end{bmatrix}$$

$$x = \begin{bmatrix} 3444444.44 \\ 122222.22 \end{bmatrix}$$

Network Flow: The traffic flow in Janakpur on several one-way streets on a regular early afternoon is shown in the accompanying figure. Vehicles per hour are used to express the data. Provide the general flux model of the network as follows:

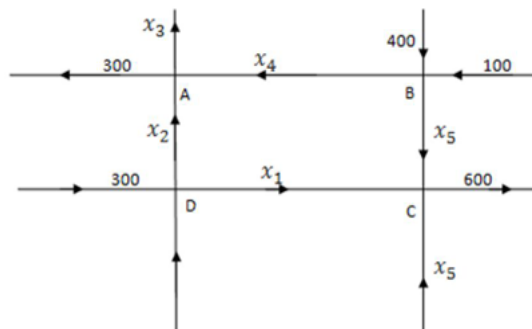


Figure: Traffic flux

Table 1: Demonstrates that at the intersection, the total inflow and outflow are equal.

Junction	influx	outflux
A	$x_2 + x_4$	$300 + x_3$
B	500	$x_4 + x_5$
C	$x_1 + x_5$	600
D	800	$x_1 + x_2$

Total inflow = Total outflow

$$\Rightarrow x_2 + x_4 + 500 + x_1 + x_5 + 800 = 300 + x_3 + x_4 + x_5 + 600 + x_1 + x_2$$

$$\Rightarrow x_3 = 400$$

The following requirements are expressed as follows for a simultaneous solution:

$$x_2 + x_4 - x_3 = 300,$$

$$x_4 + x_5 = 500,$$

$$x_1 + x_5 = 600,$$

$$x_1 + x_2 = 800,$$

$$x_3 = 400$$

Vehicles are represented by the integers x_1 , x_2 , x_3 , x_4 , and x_5 , which must all be whole numbers. A system of 5 variable linear Diophantine equations is the one that was generated above. The following is the result of the simplification solution:

$$x_1 = 600 - x_5$$

$$x_2 = 200 + x_5$$

$$x_3 = 400$$

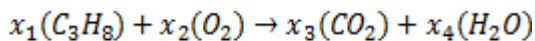
$$x_4 = 500 - x_5$$

$$x_5 = \text{freevariable}$$

Since the streets in the problem are one-way, no variable in the Network Branch may have a negative flow, which in the above model would equate to a flow in the other direction. This information places certain limitations on the variables that could be chosen. For Example, x_4 is less than 500 (because x_4 cannot be minus). The selection of x_5 between 0 and 500 determines the solutions, which is whole number.

Balancing chemical equations:

The following equation describes how propane (C_3H_8) reacts with oxygen (O_2) during combustion to produce carbon dioxide (CO_2) and water (H_2O)



Solve for $x_1, x_2, x_3, x_4 \in \mathbb{N} \cup \{0\}$ to equalize the amount of carbon, hydrogen, and oxygen atoms on the left and the equivalent atoms on the right. Since these equations include three different types of atoms, we generate a vector in $220 \mathbb{R}^3$ for each reactant of the form.

$$\begin{bmatrix} \text{carbon} \\ \text{hydrogen} \\ \text{oxygen} \end{bmatrix}$$

We get,

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

The general answer is obtained by row reduction of the resulting matrix

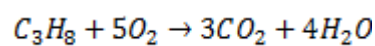
$$x_1 = \frac{1}{4}x_4$$

$$x_2 = \frac{5}{4}x_4$$

$$x_3 = \frac{3}{4}x_4$$

$$x_4 = \text{freevariable}$$

Let $x_4 = 4$ since a chemical equation's coefficients must be integers. In this instance, the Balanced Equation becomes $x_1=1$, $x_2=5$, and $x_3=3$



Note: If $x_4 = 4K$, $K \in \mathbb{N}$, the equation would also be balanced.

In contrast, chemists would rather employ a balanced equation with coefficients that are as little as possible in whole integers. Thus, the best option available is $x_4 = 4$. The equation $x^2 + y^2 = z^2$ defines a Pythagorean triple as a collection of 3 numbers, x , y , and z , such that $x^2 + y^2 = z^2$. If $\gcd(x, y, z) = 1$, the triple is considered primal.

Result: Every solution to the Pythagorean equation $x^2 + y^2 = z^2$ that satisfies the requirements

$$\gcd(x, y, z) = 1, \quad 2 \nmid x, x > 0, y > 0, z > 0$$

and given by the following formulas: for integers $s > t > 0$ such that $\gcd(s, t) = 1$ and $s \not\equiv t \pmod{2}$.

$$x = 2st, y = s^2 - t^2, z = s^2 + t^2$$

According to Fermat, the man known as the Father of Number Theory, one cannot write a cube as the sum of two cubes, $\frac{1}{4^{\text{th}}}$ power as the sum of $\frac{2}{4^{\text{th}}}$ powers, and so forth. All he was saying was that there is no integer solution to the Diophantine Equation (Non-Linear) $x^n + y^n = z^n$ if $n > 2$. Apart from the simple answer where one or more of the variables are zero. Fermat's Last Theorem is the name given to the quotation that was just mentioned. We shall now examine various applications of Pythagorean triples in real life.

Ladder Problem: Walls need to be painted by painters. A 30 feet ladder is lying against the wall, with its base situated 17 feet from the building's base. The height above the floor at which the ladder meets the wall.

By Pythagorean triples,

$$x^2 = 30^2 - 17^2 \Rightarrow \sqrt{611}_{\text{feet}}$$

Conclusion

The integral solutions of Diophantine equations make them highly useful in a wide range of real-world situations. This study examined Diophantine equations' theoretical foundations as well as their real-world applications, which included chemical balance, intricate traffic flow models, and age-related issues. This examination of these equations demonstrates how useful they are for solving problems, especially when integer solutions are needed. In addition, the study highlights how these mathematical concepts underlie common phenomena by looking at well-known outcomes like Pythagorean triples and Fermat's Last Theorem. This provides both abstract understanding and concrete application instances in industries like commerce, chemistry, and logistics. In addition to improving knowledge of mathematical theory, diophantine equations offer useful resources for solving problems in the real world.

REFERENCES

- [1] David M.Burton, Elementary Number theory, 7th Edition, Tata McGraw Hill.
- [2] J.K.Thukral, Business Maths and Statistics, Maximax publishing house, (2015).
- [3] David C.Lay, Linear Algebra and its applications, 3rd Edition, Pearsons publishing house.
- [4] Das, R., Mahato, A. K., & Sahani, S. K. (2024). Study on Computer Applications of Number Theory in Cryptography”. CENTRAL ASIAN JOURNAL OF MATHEMATICAL THEORY AND COMPUTER SCIENCES, Volume 5(3), 320-328.
- [5] Mahato, A. K, Sahani, S. K, & Das, R. (2024). Study on Computer Applications of Number Theory: New Study on Congruence in Mathematical Modeling. International Journal of Mathematical Theory and Computer Science Issues, Volume 1(2), 9–14.
- [6] Mordell, L. J. (1969). *Diophantine equations*. Academic Press.

- [7] Niven, I., Zuckerman, H. S., & Montgomery, H. L. (1991). *An introduction to the theory of numbers* (5th ed.). John Wiley & Sons.
- [8] Cohen, H. (2007). *Number theory: Volume I: Tools and Diophantine equations*. Springer.
- [9] Baker, A. (1970). *Transcendental number theory*. Cambridge University Press.
- [10] Barbeau, E. J. (2003). *Pell's equation*. Springer.
- [11] Schlickewei, H. P., Schmidt, W. M., & Evertse, J. H. (2000). *Linear equations in variables which lie in a multiplicative group*. Acta Arithmetica, 93(1), 33-49.
- [12] Cohn, H. (1980). *A classical invitation to algebraic numbers and class fields*. Springer.
- [13] Davenport, H. (2005). *The higher arithmetic: An introduction to the theory of numbers* (8th ed.). Cambridge University Press.